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Algebraic Structures

In mathematics, **algebraic structures** are sets equipped with one or more operations that satisfy certain axioms. The most common algebraic structures are groups, rings, and fields.

Definition 1 Group

A group is a set G equipped with a single binary operation $*$, denoted often by $(G,*)$ and satisfies four axioms:

1. **Closure:** For every $a, b \in G$, the result of $a * b$ is also in G.
2. **Associativity:** For every $a, b, c \in G$, $(a * b) * c = a * (b * c)$.
3. **Identity Element:** There exists an element $e \in G$ such that for every $a \in G$, $e * a = a * e = a$.
4. **Inverse Element:** For every $a \in G$, there exists an element $a' \in G$ such that $a * a' = a' * a = e$.

Remark 1

A group is called **commutative** or **abelian** if the operation $*$ satisfies the commutative property, meaning that for any elements a and b in G, $a * b = b * a$.

Definition 2 Ring

A ring is a set R equipped with two binary operations: $*$ and $\perp$ denoted by $(R,*,\perp)$ that satisfies the following axioms:

1. **$(R,*)$ is an Abelian group.**
2. **Associativity of $\perp$:** For every $a, b, c \in R$, $(a \perp b) \perp c = a \perp (b \perp c)$.
3. **Distributive Properties:** $\perp$ distributes over $*$, i.e., for every $a, b, c \in R$, $a \perp (b * c) = (a \perp b) * (a \perp c)$ and $(a * b) \perp c = (a \perp c) * (b \perp c)$

Remark 2

1. A ring is called **commutative** if the operation $\perp$ is commutative, meaning that for any elements a and b in R, $a \perp b = b \perp a$.
2. A ring is called **unital** (or **ring with unity**) if it has an identity element e for the operation $\perp$, meaning that for every element a in R, $a \perp e_2 = e_2 \perp a = a$

Definition 3 Field

A field is a set F equipped with two binary operations: $*$ and $\perp$ denoted by $(F,*,\perp)$ and satisfies the following axioms:

1. **$(F,*,\perp)$ is an unital ring.**
2. **Every element (except e_1) has a symmetric element.**
   For all $a \in F \setminus e_1$, there exists $a' \in F$ such that $a \perp a' = a' \perp x = e_2$.

Remark 3

A field is called **commutative field** if the operation $\perp$ is commutative, meaning that for any elements a and b in F,$a \perp b = b \perp a$.

❖ Introduction to Vector Spaces

A vector space (also called a linear space) over a field F is a set V along with two operations: vector addition and scalar multiplication on V. The elements of V are called vectors, and the elements of a field F are called scalars.

Definition 4 Vector Space

Let F be a field, a vector space V over a field F is a non-empty set that must satisfy the following axioms:

1. **Closure under Addition**: For every $u, v \in V$, the sum $u + v \in V$.
2. **Closure under Scalar Multiplication**: For every $v \in V$ and scalar $a \in F$, the product $a \times v \in V$.
3. **Commutativity of Addition**: For every $u, v \in V$, $u + v = v + u$.
4. **Associativity of Addition**: For every $u, v, w \backslash in\, V$, $(u + v) + w = u + (v + w)$.
5. **Existence of Additive Identity**: $\exists\, 0 \in V$ such that $v + 0 = v$ for all $v \in V$.
6. **Existence of Additive Inverse**: $\forall\, v \in V$, $\exists\, -v \in V$ such that $v + (-v) = 0$.
7. **Associativity of Scalar Multiplication**: For every $a, b \in F$ and $v \in V$, $a(bv) = (ab)v$.
8. **Existence of Multiplicative Identity**: $\exists\, 1 \in F$ such that $1 \times v = v$ for all $v \in V$.

9. **Distributivity of Scalar Multiplication with Respect to Vector Addition**: For every $a \in F$ and $u, v \in V$, $a(u + v) = au + av$.

10. **Distributivity of Scalar Multiplication with Respect to Field Addition**: For every $a, b \in F$ and $v \in V$, $(a + b)v = av + bv$.

Example 1

1. **Real Numbers** $(\mathbb{R})$: The set of all real numbers forms a vector space under standard addition and scalar multiplication.
2. **Coordinate Space** $(\mathbb{R}^n)$: The set of all (n)-tuples of real numbers forms an (n)-dimensional vector space.
3. **Matrices**: The set of all $(m \times n)$ matrices form a vector space.
4. **Polynomials**: The set of all polynomials with real coefficients forms a vector space.
5. **Functions**: The set of all continuous functions from $\mathbb{R} \to \mathbb{R}$ forms a vector space.

Remark 4

If F is a scalar field, then F is a vector space over F itself.

❖ Subspaces

A subspace is a subset of a vector space that is itself a vector space under the same operations of addition and scalar multiplication.

Definition 5 Subspace

A subset W of a vector space V is called a subspace if it satisfies the following conditions:

1. The zero vector of V is in W.
2. ***W* is closed under vector addition**: if u and v are in W, then $u + v$ is also in W.
3. ***W* is closed under scalar multiplication**: if u is in W and a is a scalar, then $a \times u$ is also in W.

Example 2

1. $\mathbb{Q}$ is a subspace of $\mathbb{R}$.
2. The set $F = \{(x, y) \in \mathbb{R}^2 \mid x + y = 0\}$ is a vector subspace of $\mathbb{R}^2$.

Remark 5

If V is a F- vector space and W is a subspace of V, then W is itself a F- vector space for the laws induced by V.

Definition 6 Intersection of Subspaces

The intersection of two subspaces V_1 and V_2 of a vector space V is the set of all vectors that are in both V_1 and V_2 . It is denoted by $V_1 \cap V_2$ and is itself a subspace of V.

Definition 7 Sum of Subspaces

The sum of two subspaces V_1 and V_2 of a vector space V is the set of all possible sums of elements from V_1 and V_2. It is denoted by $V_1 + V_2$ and is also a subspace of V.

Suppose $V_1, \dots, V_m$ are subspaces of V.
Then $V_1 + \cdots + V_m = \{v_1 + \cdots + v_m : v_k \in V_k, k \in \{1, \dots, m\}\}$

❖ Linear Combinations and Linear Independence

Linear Combinations

A linear combination of vectors involves expressing one vector as a weighted sum of other vectors.

Definition 8 Linear Combination

Given an integer ($n \geq 1$) and vectors $v_1, v_2, \dots, v_n$ in a vector space V.

A linear combination of these vectors is an expression of the form: $\boldsymbol{u} = \boldsymbol{\lambda_1 v_1} + \boldsymbol{\lambda_2 v_2} + \cdots + \boldsymbol{\lambda_n v_n}$ where $\lambda_1, \lambda_2, \dots, \lambda_n$ are elements of the field F called the coefficients of the linear combination.

Linear Independence

A set of vectors is linearly independent if and only if no nontrivial linear combination of these vectors equals the zero vector. Each vector in the set cannot be expressed as a linear combination of the others.

Definition 9 Linear Independence

Given a list $v_1, v_2, \dots, v_p$ in a vector space V.

This list is called linearly independent if the unique solution to the equation: $\boldsymbol{\lambda_1 v_1} + \boldsymbol{\lambda_2 v_2} + \cdots + \boldsymbol{\lambda_p v_p} = \boldsymbol{0}$ is the trivial solution $\boldsymbol{\lambda_1} = \boldsymbol{\lambda_2} = \cdots = \boldsymbol{\lambda_p} = \boldsymbol{0}$.

Mathematically: $\boldsymbol{\lambda_1 v_1} + \cdots + \boldsymbol{\lambda_p v_p} = \boldsymbol{0} \Leftrightarrow \forall \boldsymbol{i} \in \{\boldsymbol{1}, \dots, \boldsymbol{p}\} : \boldsymbol{\lambda_i} = \boldsymbol{0}$

The empty list ( ) is also declared to be linearly independent.

Definition 10 Linear dependence

A list of vectors in V is called **linearly dependent** if it is not linearly independent.

In other words, a list $v_1, v_2, \dots, v_p$ of vectors in V is linearly dependent if and only if there exists at least one vector $v_i, i \in \{1, .., p\}$ where v_i is a linear combination of the other vectors, i.e. $v_i \in span\ (v_1, \dots, v_k)\ with\ k < p$.

❖ Bases and Dimension of a Vector Space

Definition 11 Span

The set of all linear combinations of a list of vectors $v_1, \dots, v_m$ in V is called the **span** of $v_1, \dots, v_m$, denoted by $span\ (v_1, \dots, v_m)$.

$\boldsymbol{span\ (v_1, \dots, v_m)} = \{\boldsymbol{a_1 v_1} + \cdots + \boldsymbol{a_m v_m} : \boldsymbol{a_1}, \dots, \boldsymbol{a_m} \in \boldsymbol{F}\}$.

The span of the empty list ( ) is defined to be $\{0\}$.

If $span\ (v_1, \dots, v_m)$ equals V, we say that the list $v_1, \dots, v_m$ **spans** V.

Mathematically: $\forall\, \boldsymbol{v} \in \boldsymbol{V}, \exists\, \boldsymbol{\lambda_1}, \dots, \boldsymbol{\lambda_p} \in \boldsymbol{F} : \boldsymbol{v} = \boldsymbol{\lambda_1 v_1} + \cdots + \boldsymbol{\lambda_m v_m}$

Example 3

$(17, -4, 2) \in span\ \big((2, 1, -3), (1, -2, 4)\big)$.

$(17, -4, 5) \notin span\ ((2, 1, -3), (1, -2, 4))$.

Example 4

The list $(1, 0);\ (0, 1);\ (1, 1)$ spans $\mathbb{R}^2$

Definition 12 Basis

A basis of V is a list of vectors in V that is linearly independent and spans V.

In other word, a list $v_1, \dots, v_n$ of vectors in V is a basis of V if and only if every $v \in V$ can be written uniquely in the form $\boldsymbol{v} = \boldsymbol{\lambda_1 v_1} + \cdots + \boldsymbol{\lambda_n v_n}$ where $\lambda_1, \dots, \lambda_n \in F$.

Mathematically: $\forall\, \boldsymbol{v} \in \boldsymbol{E}, \exists!\ \boldsymbol{\lambda_1}, \dots, \boldsymbol{\lambda_n} \in \boldsymbol{F} : \boldsymbol{v} = \boldsymbol{\lambda_1 v_1} + \cdots + \boldsymbol{\lambda_n v_n}$

Remark 6

1. Every spanning list in a vector space can be reduced to a basis of the vector space.
2. Every linearly independent list of vectors in a finite-dimensional vector space can be extended to a basis of the vector space.

Definition 13 Dimension

The dimension of a **finite-dimensional** vector space denoted by $\boldsymbol{dim\, V}$ is the length of any basis of the vector space.

The dimension of the vector space $\{0\}$ is 0.

Remark 7

If V is finite-dimensional and U is a subspace of V, then $\dim U \le \dim V$.

❖ Linear Maps

Definition 14 Linear Map

A linear map or linear transformation from V to W is a function $T : V \to W$ with the following properties.

1. **Additivity**

$T(u + v) = Tu + Tv\ for\ all\ u, v \in V.$

2. **homogeneity**

$T(\lambda v) = \lambda(Tv)\ for\ all\ \lambda \in F\ and\ all\ v \in V.$

Remark 8

1. If T is a linear map from V to W, then $T(0) = 0$.
2. The set of linear maps from V to W is denoted by $\mathcal{L}(V, W)$.
3. The set of linear maps from V to V is denoted by $\mathcal{L}(V)$.
4. $\mathcal{L}(V, W)$ is a vector space with the operations of addition and scalar multiplication.

❖ Kernel and Image Spaces

Definition 15 Kernel

For $T \in \mathcal{L}(V, W)$, the kernel of T, denoted by $\boldsymbol{ker(T)}$, is the subset of V consisting of those vectors that T maps to 0.

$\boldsymbol{Ker\ (T) = \{v \in V : T(v) = 0\}}.$

Remark 9

1. $Ker\ (T)$ is called also the null space of T and denoted by $null\ T$.
2. Suppose $T \in \mathcal{L}(V, W)$. Then $Ker(T)$ is a subspace of V.
3. Let $T \in \mathcal{L}(V, W)$. Then T is injective if and only if $Ker\ (T) = \{0\}$.

Definition 16 Range

For $T \in \mathcal{L}(V, W)$, the **range** of T is the subset of W consisting of those vectors that are equal to Tv for some $v \in V$: $range\ T = \{T(v) : v \in V\}$.

Remark 10

1. Suppose $T \in \mathcal{L}(V, W)$. Then $range\ T$ is a subspace of W.

2. Let $T \in \mathcal{L}(V, W)$. Then T is surjective if and only if $range\ T = W$.
3. If V is finite-dimensional, then $dim\ V = dim\ ker\ T + dim\ range\ T$.

➢ Matrices

❖ Introduction to Matrices

Definition 1

Suppose m and n are nonnegative integers. An $m \times n$ matrix A is a rectangular array of elements of F with m rows and n columns:

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \cdots & a_{2n} \\ \vdots & \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & \cdots & a_{mn} \end{pmatrix}$$

Mathematically: In mathematics, we can write $A \in \mathcal{M}_{m,n}(F)$ or $A \in F^{m,n}$, which means that A is a matrix of size $m \times n$ with elements in F.

Example 1

$A = \begin{pmatrix} 3 & -i & 2 \\ 6 & 55 & 5 \\ 14 & 10 & 0 \\ 9+15i & 0 & 5-i \end{pmatrix}$ is a matrix of size 4×3 with elements in $\mathbb{C}$.

We can also write $A \in \mathcal{M}_{4,3}(\mathbb{C})$.

Remark 1

1. The notation a_{ij} denotes the entry in row i, and column j of A.
2. A can be represented as: $A = (a_{ij})$.

❖ Operations on Matrices

In linear algebra, one relation (equality) and four operations (addition, subtraction, scalar multiplication, and matrix multiplication) are defined for matrices.

Definition 2 Equality

Let $A = (a_{ij})$ and $B = (b_{ij})$ be two matrices.

Two matrices A and B are said to be equal if:
1. A and B are of the same size.
2. $a_{ij} = b_{ij}$ for all $1 \le i \le m$ and $1 \le j \le n$.

Definition 3 Addition

Let $A = (a_{ij})$ and $B = (b_{ij})$ be two matrices $1 \le i \le m$ and $1 \le j \le n$.

$A + B$ is the matrix denoted by C such that $C = (c_{ij})$ where $c_{ij} = a_{ij} + b_{ij}$ for all $1 \le i \le m$ and $1 \le j \le n$.

Addition is defined if the two matrices have the same number of rows and the same number of columns.

Example 2

$$\begin{pmatrix} 1 & 17 & 23 & 6 \\ 6 & 5 & 33 & 8 \end{pmatrix} + \begin{pmatrix} 18 & 36 & 7 & 10 \\ 20 & 4 & 6 & 29 \end{pmatrix} = \begin{pmatrix} 19 & 53 & 30 & 16 \\ 26 & 9 & 39 & 37 \end{pmatrix}$$

Remark 2

1. The addition of matrices is commutative, meaning $A + B = B + A$.
2. The addition of matrices is associative, meaning $(A + B) + C = A + (B + C)$.

Definition 4 Subtraction

Let $A = (a_{ij})$ and $B = (b_{ij})$ be two matrices $1 \leq i \leq m$ and $1 \leq j \leq n$.

$A - B$ is the matrix denoted by C such that $C = (c_{ij})$ where $c_{ij} = a_{ij} - b_{ij}$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$.

As addition subtraction is defined if the two matrices have the same number of rows and the same number of columns.

Example 3

$$\begin{pmatrix} 15 & -9 \\ 8 & 10 \end{pmatrix} - \begin{pmatrix} -1 & 2 \\ 7 & 10 \end{pmatrix} = \begin{pmatrix} 16 & -11 \\ 1 & 0 \end{pmatrix}$$

Definition 5 Scalar Multiplication

Let $A = (a_{ij})$ be a matrix of size $m \times n$ and α be a scalar.

The product of a scalar α by a matrix A, denoted as αA, is defined to be the matrix obtained by multiplying each element of A by α.

$\alpha A = (\alpha \times a_{ij})$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$.

Example 4

Let $A = \begin{pmatrix} 12 & 5 & 2 \\ 1 & 17 & 9 \end{pmatrix}$ be a matrix and $\alpha = 5$ be a scalar.

$$\alpha A = 5 \times \begin{pmatrix} 12 & 5 & 2 \\ 1 & 17 & 9 \end{pmatrix} = \begin{pmatrix} 5 \times 12 & 5 \times 5 & 5 \times 2 \\ 5 \times 1 & 5 \times 17 & 5 \times 9 \end{pmatrix} = \begin{pmatrix} 60 & 25 & 10 \\ 5 & 85 & 45 \end{pmatrix}$$

Property 1

Let A and B be two matrices of the same size and let α and β be two scalars.

1. $\alpha A = A\alpha$.
2. $\alpha(\beta A) = (\alpha\beta)A$.
3. $(\alpha + \beta)A = \alpha A + \beta A$.
4. $\alpha(A + B) = \alpha A + \alpha B$.

Definition 6 Matrix Multiplication

Let $A = (a_{ik})$ be a matrix of size $m \times l$ and $B = (b_{kj})$ be a matrix of size $l \times n$.

The multiplication of two matrices A and B is defined if the number of columns of the first matrix equals the number of rows of the second matrix.

The product $A \times B$ is a matrix C of size $m \times n$, where the elements c_{ij} of the matrix C can be calculated using the following formula: $c_{ij} = \sum_{k=1}^{l} a_{ik} \times b_{kj}$.

$$\begin{pmatrix} a_{11} & \cdots & a_{1l} \\ a_{21} & \cdots & a_{2l} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{ml} \end{pmatrix}_{m \times l} \times \begin{pmatrix} b_{11} & b_{12} & \cdots & \cdots & b_{1n} \\ \vdots & \vdots & & & \vdots \\ b_{l1} & b_{l2} & \cdots & \cdots & b_{ln} \end{pmatrix}_{l \times n} = \begin{pmatrix} c_{11} & c_{12} & \cdots & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & \cdots & c_{2n} \\ \vdots & \vdots & & & \vdots \\ c_{m1} & c_{m2} & \cdots & \cdots & c_{mn} \end{pmatrix}_{m \times n}$$

Example 5

Let $A = \begin{pmatrix} 1 & 2 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 5 \\ -3 & 2 \\ 7 & 10 \end{pmatrix}$ be two matrices.

$A \times B = \begin{pmatrix} -6 & 9 \end{pmatrix}$

Example 6

Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ and $B = \begin{pmatrix} 10 & -2 & 7 \\ 0 & 0 & -1 \end{pmatrix}$ be two matrices.

This product is impossible, because the number of columns of the first matrix (A) does not equal to the number of rows of the second matrix (B).

Property 2

Let A, B and C be three matrices where multiplication is possible.

1. $(AB)C = A(BC)$.
2. $A(B + C) = AB + AC$.
3. $(B + C)A = BA + CA$.

Remark 3

1. In general, matrix multiplication is not commutative $AB \neq BA$.
2. If $AB = 0 \nRightarrow A = 0 \vee B = 0$.

Example 7

Let $A = \begin{pmatrix} 5 & 1 \\ 2 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ be two matrices.

$A \times B = \begin{pmatrix} 1 & 6 \\ 0 & 2 \end{pmatrix}$ but $B \times A = \begin{pmatrix} 2 & 0 \\ 7 & 1 \end{pmatrix}$

Example 8

Let $A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$ be two matrices.

$A \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \wedge B \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ but $A \times B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

❖ Types of Matrices

Definition 7 Zero Matrix

Let $A = (a_{ij})$ be a matrix of size $m \times n$.

A is a zero matrix $(A = 0)$, if all its elements are zero.

$$A = \begin{pmatrix} 0 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & \cdots & \cdots & 0 \end{pmatrix}$$

Mathematically: $\forall a_{ij} \in A, a_{ij} = 0 \; with \; 1 \leq i \leq m \; et \; 1 \leq j \leq n$

Example 9

Let $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

A is a zero matrix.

Definition 8 Square Matrix

Let $A = (a_{ij})$ be a matrix.

A is a square matrix if it has an equal number of rows and columns. In other words, if a matrix has dimensions $n \times n$, where n is a positive integer, then it is called a square matrix.

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$

The elements $a_{11}, a_{22}, \dots, a_{nn}$ are called the elements of the main diagonal.

Example 10

Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$

A is a square matrix because it has the same number of rows and columns.

Remark 4

If A is a square matrix we can say: A is of size $n \times n$ or of order n.

Definition 9 Identity Matrix

Let $A = (a_{ij})$ be a square matrix of size $n \times n$.

An identity matrix, often denoted by I or I_n , is a square matrix with 1s on the main diagonal and 0s elsewhere.

$$I = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1 \end{pmatrix}$$

Mathematically: $\forall\, 1 \le i \le n, et\ 1 \le j \le n,\ a_{ij} = \begin{cases} 1\ si\ i = j \\ 0\ si\ i \ne j \end{cases}$

Example 11

Let $I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

I is an identity matrix for matrices of order 3.

Definition 10 Diagonal Matrix

Let $A = (a_{ij})$ be a square matrix of size $n \times n$.

We say that A is a diagonal matrix if its elements below and above the main diagonal are all zeros.

$$: A = \begin{pmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & a_{nn} \end{pmatrix}$$

Mathematically: $\forall\, 1 \le i \le n, et\ 1 \le j \le n,\ i \ne j \implies a_{ij} = 0$

Example 12

Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

A is a diagonal matrix.

I the identity matrix is also a diagonal matrix.

Example 13

Let $I = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1 \end{pmatrix}$

The identity matrix is also a diagonal matrix.

Definition 11 Upper Triangular Matrix

Let $A = (a_{ij})$ be a square matrix of size $n \times n$.

We say that A is an upper triangular matrix if its elements below the main diagonal are all zeros.

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn} \end{pmatrix}$$

Mathematically: $\forall\ 1 \le i \le n, and\ 1 \le j \le n\ if\ i > j \implies a_{ij} = 0$

Example 14

Let $A = \begin{pmatrix} 1 & 0 & 5 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}$

A is an upper triangular matrix.

Definition 12 Lower Triangular Matrix

Let $A = (a_{ij})$ be a square matrix of size $n \times n$.

We say that A is a lower triangular matrix if its elements above the main diagonal are all zeros.

$$A = \begin{pmatrix} a_{11} & 0 & \dots & 0 \\ a_{21} & a_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$

Mathematically: $\forall\ 1 \le i \le n, and\ 1 \le j \le n, if\ i < j \implies a_{ij} = 0$

Example 15

Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 2 & 0 \\ 0 & 4 & 2 \end{pmatrix}$

A is a lower triangular matrix.

Definition 13 Transpose of a Matrix

Let $A = (a_{ij})$ be a matrix of size $m \times n$.

The transpose of a matrix A, denoted by A^t, is the matrix obtained from A by interchanging rows and columns. Specifically, if A is an $m \times n$ matrix, then A^t is the $n \times m$ matrix.

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & \dots & a_{2n} \\ \vdots & \vdots & & & \vdots \\ a_{m1} & a_{m2} & \dots & \dots & a_{mn} \end{pmatrix} \Leftrightarrow A^t = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{pmatrix}$$

Example 16

Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \Rightarrow A^t = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}$

Let $B = \begin{pmatrix} -1 & 8 & 7 \\ 3 & 5 & 6 \\ 0 & 2 & -4 \end{pmatrix} \Rightarrow B^t = \begin{pmatrix} -1 & 3 & 0 \\ 8 & 5 & 2 \\ 7 & 6 & -4 \end{pmatrix}$

Property 3

Let A and B be two matrices of size $m \times n$, C a matrix of size $n \times l$ and $\alpha \in F$.

1. $(A^t)^t = A$.
2. $(\alpha A)^t = \alpha A^t$.
3. $(A + B)^t = A^t + B^t$.
4. $(AC)^t = C^t A^t$

Definition 14 Symmetric Matrix

Let $A = (a_{ij})$ be a square matrix of order n.

A is a symmetric matrix if it is equal to its transpose.

Mathematically: $A^t = A$

Example 17

Let $A = \begin{pmatrix} 1 & -3 & 0 \\ -3 & 2 & 5 \\ 0 & 5 & 3 \end{pmatrix}$

A is a symmetric matrix because: $A^t = \begin{pmatrix} 1 & -3 & 0 \\ -3 & 2 & 5 \\ 0 & 5 & 3 \end{pmatrix} = A$

Definition 15 Asymmetric Matrix

Let $A = (a_{ij})$ be a square matrix of order n.

A is asymmetric matrix if $A^t = -A$.

Example 18

Let $A = \begin{pmatrix} 0 & -i & 5 \\ i & 0 & 3 \\ -5 & -3 & 0 \end{pmatrix}$

A is an Asymmetric matrix because: $A^t = \begin{pmatrix} 0 & i & -5 \\ -i & 0 & -3 \\ 5 & 3 & 0 \end{pmatrix} = -A$

Remark 5

An asymmetric matrix is also called: anti-symmetric matrix or skew-symmetric matrix.

❖ Trace and Determinant of a Matrix

Definition 16 Trace of a Matrix

Let $A = (a_{ij})$ be a square matrix of order n.

The trace of A, denoted by $tr(A)$, is defined to be the sum of the diagonal entries of A.

Mathematically: $tr(A) = \sum_{i=1}^{n} a_{ii} = a_{11} + a_{22} + \cdots + a_{nn}$.

Example 19

Let $A = \begin{pmatrix} 1 & 6 & 1 \\ 4 & -7 & 3 \\ 9 & 5 & 8 \end{pmatrix} \Rightarrow tr(A) = 1 + (-7) + 8 = 2.$

Property 4

Let A and B be two square matrices of order n.

1. $tr(A + B) = tr(A) + tr(B).$
2. $tr(\alpha A) = \alpha . tr(A)\ with\ \alpha \in F.$
3. $tr(A^T) = tr(A).$
4. $tr(AB) = tr(BA).$

Definition 17 Determinant of a Matrix

Let $A = (a_{ij})$ be a square matrix of order n.

The determinant of a square matrix A, denoted as $det(A)$, is a scalar calculated recursively as follows:

1. For $n = 1,\ det(A) = a_{11}.$
2. For $n = 2,\ A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \Leftrightarrow \det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} \times a_{22} - a_{21} \times a_{12}.$
3. For $n > 2,\ \det(A) = \sum_{j=1}^{n} (-1)^{i+j} \times a_{ij} \times \det(A_{ij})$ with $1 \leq i \leq n.$

Where A_{ij} is a matrix obtained from A by deleting the i^th row and the j^{th} column of A.

Example 20

$$A = \begin{pmatrix} 2 & 5 & -4 \\ 6 & 0 & 1 \\ 9 & 10 & 4 \end{pmatrix} \Leftrightarrow \begin{vmatrix} 2 & 5 & -4 \\ 6 & 0 & 1 \\ 9 & 10 & 4 \end{vmatrix} = 2 \times \begin{vmatrix} 0 & 1 \\ 10 & 4 \end{vmatrix} - 5 \begin{vmatrix} 6 & 1 \\ 9 & 4 \end{vmatrix} + (-4) \begin{vmatrix} 6 & 0 \\ 9 & 10 \end{vmatrix}$$

$$= 2 \times (0 \times 4 - 10 \times 1) - 5 \times (6 \times 4 - 9 \times 1) + (-4)(6 \times 10 - 9 \times 0) = -335$$

❖ Inverse of a Matrix

Definition 18 Inverse of a Matrix

Let $A = (a_{ij})$ be a square matrix of order n.

The inverse of a square matrix A is a square matrix B such that $A \times B = B \times A = I_n$.

The inverse of A is defined by: $A^{-1} = \frac{1}{\det(A)} \times C^t$.

where $C = (-1)^{i+j} M_{ij}$ with $1 \leq i, j \leq n$ is called the cofactor matrix of A. M_{ij} is the determinant of the $(n-1) \times (n-1)$ matrix obtained by removing the i^{th} row and the j^{th} column from A.

Remark 6

1. The inverse is defined only for square matrices.
2. $det(A) \neq 0 \Leftrightarrow A^{-1}\ exists.$
3. I_n is invertible, and its inverse is I_n itself.

Property 5

Let A be an invertible matrix.

1. If A is invertible, the inverse is unique.
2. A^{-1} is also invertible, and $(A^{-1})^{-1} = A$

Example 21

Let $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \Leftrightarrow A^{-1} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$

❖ Eigenvalues and Eigenvectors

Definition 19 Eigenvalues & Eigenvectors

Let $A = (a_{ij})$ be a square matrix of order n.

λ is called an eigenvalue of the matrix A if there exists a non-zero vector $X \in F^n$ such that $\boldsymbol{AX} = \boldsymbol{\lambda X}$.
The vector X is called the eigenvector of A associated with the eigenvalue λ.
Mathematically: $\lambda \text{ is an eigenvalue of } A \iff \exists X \in (F^n)^* : AX = \lambda X.$
$X \text{ is an eigenvector of } A \iff \exists \lambda \in F : AX = \lambda X.$

Example 22

Let $A = \begin{pmatrix} 6 & 3 \\ 5 & 8 \end{pmatrix}$
Show that $\lambda = 3$ is an eigenvalue of A.
We know that λ *is an eigenvalue of* $A \iff \exists X \in (F^n)^* : AX = \lambda X$.
So $\begin{pmatrix} 6 & 3 \\ 5 & 8 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -3\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{cases} 6x + 3y = 3x \\ 5x + 8y = 3y \end{cases} \Rightarrow y = -x \Rightarrow \left\{\begin{pmatrix} x \\ -x \end{pmatrix} \text{such that } x \in \mathbb{R}\right\}$
Thus, the solutions are generated by the vector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, which is the eigenvector associated with the eigenvalue $\lambda = 3$.

Example 23

Let $A = \begin{pmatrix} 6 & 3 \\ 5 & 8 \end{pmatrix}$
Is $\begin{pmatrix} 3 \\ 5 \end{pmatrix}$ an eigenvector of A.
We know that X *is an eigenvector of* $A \iff \exists \lambda \in F : AX = \lambda X$.
So $\begin{pmatrix} 6 & 3 \\ 5 & 8 \end{pmatrix}\begin{pmatrix} 3 \\ 5 \end{pmatrix} = \lambda\begin{pmatrix} 3 \\ 5 \end{pmatrix} = 0 \Rightarrow \begin{cases} 33 = 3\lambda \\ 55 = 3\lambda \end{cases} \Rightarrow \lambda = 11$
Thus, $\begin{pmatrix} 3 \\ 5 \end{pmatrix}$ is an eigenvector of A and its associated eigenvalue is $\lambda = 11$.

Property 6

Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be the eigenvalues associated with a given matrix A
1. $\lambda_1 + \lambda_2 + \dots + \lambda_n = tr(A)$.
2. $\lambda_1 \times \lambda_2 \times \dots \times \lambda_n = det(A)$.

Definition 20 Characteristic Polynomial

Let $A = (a_{ij})$ be a square matrix of order n.

The polynomial $p(\lambda) = det(A - \lambda I)$ is called the characteristic polynomial of A.

Example 24

Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$
$p(\lambda) = det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 \\ 3 & 4-\lambda \end{vmatrix} = (1-\lambda)(4-\lambda) - 6 = \lambda^2 - 5\lambda - 2.$

Property 7

Let P be the characteristic polynomial of a given matrix A.
The eigenvalues of A are the roots of the characteristic polynomial of A.
Mathematically: $\lambda \text{ is an eigenvalue of } A \Rightarrow p(\lambda) = 0.$

Example 25

Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $p(\lambda) = \lambda^2 - 5\lambda - 2$.
Find all eigenvalues of A.

$p(\lambda) = \lambda^2 - 5\lambda - 2 = 0 \Longrightarrow \lambda = \frac{5+\sqrt{33}}{2}\, or\, \lambda = \frac{5-\sqrt{33}}{2}$.

Thus, the eigenvalues of A are $\lambda_1 = \frac{5+\sqrt{33}}{2}$ *and* $\lambda_2 = \frac{5-\sqrt{33}}{2}$.

Property 8

If A is a square matrix of order n then it has at most n eigenvalues.

❖ Similar Matrices

Definition 21 Similar Matrices

Let A and B be two square matrices in $M_n(F)$.

We say that matrix B is similar to matrix A, or that A and B are similar, if there exists an invertible matrix $P \in M_n(F)$ such that $B = P^{-1} A P$.

Example 26

Let A and B be two square matrices in $M_2(\mathbb{R})$

$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 15 & 2 \\ -98 & -13 \end{pmatrix}$.

A and B are similar, because there exists an invertible matrix $P = \begin{pmatrix} 1 & 0 \\ 7 & 1 \end{pmatrix} \in M_2(\mathbb{R})$ such that $B = P^{-1} A P$

Property 9

If A and B are similar, then they have the same eigenvalues.

❖ Special Types of Matrices

Definition 22 Positive-Definite Matrix

Let $A = (a_{ij})$ be a real square matrix of order n.

A is said to be a positive-definite matrix if $v^t A v > 0$ for all non-zero $v \in \mathbb{R}^n$.

Mathematically: $A\ positive\ definite \Leftrightarrow v^t A v > 0, \forall v \in \mathbb{R}^n \backslash \{0\}$

Example 27

Let $A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$

A is a positive definite matrix.

Property 10

A is positive definite if and only if all of its eigenvalues are positive.

Definition 23 Positive Semi-Definite Matrix

Let $A = (a_{ij})$ be a real square matrix of order n.

A is said to be a positive semi-definite matrix if $v^t A v \geq 0$ for all non-zero $v \in \mathbb{R}^n$.

Mathematically: $A\ positive\ semi\ definite \Leftrightarrow v^t A v \geq 0, \forall v \in \mathbb{R}^n \backslash \{0\}$

Example 28

Let $A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$

A is a positive semi-definite matrix.

Property 11

A is positive semi-definite if and only if all of its eigenvalues are non-negative.

Definition 24 Orthogonal Matrix

Let $A = (a_{ij})$ be a real square matrix of order n.

A is called orthogonal if its inverse is equal to its transpose.

Mathematically: $A^{-1} = A^t \Longrightarrow AA^t = I$

Example 29

Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

A is an orthogonal matrix because: $A \times A^t = I$

Let $B = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

B is an orthogonal matrix because: $B \times B^t = I$

Definition 25 Involutory Matrix

Let $A = (a_{ij})$ be a square matrix of order n.

A is called involutory if it is equal to its own inverse.

Mathematically: $A^{-1} = A \Longrightarrow AA = A^2 = I$

Example 30

Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

A is an involutory matrix because: $A \times A = I$

Let $B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

B is an orthogonal matrix because: $B \times B = I$

❖ Link Between Linear Maps and Matrices

Definition 26 Link Between Linear Maps and Matrices

Let E and G be two finite-dimensional vector spaces over the field F with dimensions m and n, respectively. Let $B = (b_1, \dots, b_m)$ be a basis of E and $B' = (b'_1, \dots, b'_n)$ be a basis of G, and let $f: E \to G$ be a linear map.

The matrix of the linear map f with respect to the bases B and B' is the matrix denoted by $Mat_{B,B'}(f) = (aij) \in M_{n,m}(F)$. This matrix is composed of columns that represent the images of the basis vectors of E under f, expressed in the basis B' of G.

$$\begin{array}{c} \\ b'_1 \\ b'_2 \\ \vdots \\ b'_n \end{array} \begin{array}{c} \begin{array}{cccccc} f(b_1) & f(b_2) & \cdots & f(b_j) & \cdots & f(b_m) \end{array} \\ \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nj} & \cdots & a_{nm} \end{pmatrix} \end{array}$$

Example 31

Let f be the linear map defined by: $f(x, y, z) = (x + y - z, x - 2y + 3z)$.

Let $B = (e_1, e_2, e_3)$ be the standard basis of $\mathbb{R}^3$ and $B' = (f_1, f_2)$ be the standard basis of $\mathbb{R}^2$.

1. We need to find the images of the elements of B under f.
$e_1 = (1 \quad 0 \quad 0) \Rightarrow f(e_1) = (1 \quad 1)$
$e_2 = (0 \quad 1 \quad 0) \Rightarrow f(e_2) = (1 \quad -2)$
$e_3 = (0 \quad 0 \quad 1) \Rightarrow f(e_3) = (-1 \quad 3)$
2. We need to express these images in the basis B'
$(1 \quad 1) = f_1 + f_2$
$(1 \quad -2) = f_1 - 2f_2$
$(-1 \quad 3) = f_1 + f_2$
Thus, $Mat_{\mathcal{B},\mathcal{B}'}(f) = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -2 & 3 \end{pmatrix}$

Example 32

Let f be the linear map defined by: $f(x, y, z) = (x + y - z, x - 2y + 3z)$.
Let $\mathcal{A} = (\phi_1, \phi_2, \phi_3) = ((1 \quad 1 \quad 0), (1 \quad 0 \quad 1), (0 \quad 1 \quad 1))$ be a basis for $\mathbb{R}^3$ and $\mathcal{A}' = (\sigma_1, \sigma_2) = ((1 \quad 0), (1 \quad 1))$ be a basis for $\mathbb{R}^2$.

$\phi_1 = (1 \quad 1 \quad 0) \Rightarrow f(\phi_1) = (2 \quad -1) = 3\sigma_1 - \sigma_2$
$\phi_2 = (1 \quad 0 \quad 1) \Rightarrow f(\phi_2) = (0 \quad 4) = -4\sigma_1 + 4\sigma_2$
$\phi_3 = (0 \quad 1 \quad 1) \Rightarrow f(\phi_3) = (0 \quad 1) = -\sigma_1 + \sigma_2$

Thus, $Mat_{\mathcal{B},\mathcal{B}'}(f) = \begin{pmatrix} 3 & -4 & -1 \\ -1 & 4 & 1 \end{pmatrix}$

Remark 7

Let $f, g : E \longrightarrow F$ be two linear maps.
Let B be a basis of E and B' be a basis of F.
1. The matrix $Mat_{\mathcal{B},\mathcal{B}'}(f)$ depends on the choice of bases.
2. $Mat_{\mathcal{B},\mathcal{B}'}(f + g) = Mat_{\mathcal{B},\mathcal{B}'}(f) + Mat_{\mathcal{B},\mathcal{B}'}(g)$.
3. $Mat_{\mathcal{B},\mathcal{B}'}(\lambda f) = \lambda . Mat_{\mathcal{B},\mathcal{B}'}(f)$.

Remark 8

Let $f: E \rightarrow F$ and $g: F \rightarrow G$ be two linear maps and let B be a basis of E, B' a basis of F, and B'' a basis of G.
The matrix of the composition $f \circ g$ with respect to these bases is given by:
$Mat_{\mathcal{B},\mathcal{B}'}(f \circ g) = Mat_{\mathcal{B}',\mathcal{B}''}(g) \times Mat_{\mathcal{B},\mathcal{B}'}(f)$

➢ Norms and scalar products

❖ Norms

In linear algebra, there are two main types of norms that are commonly discussed: **vector norms** and **matrix norms**.

Vector Norms

Definition 27 Vector Norms

Let V be a vector space over the field F of scalars.
A norm on V is a function $\|.\|: V \longrightarrow \mathbb{R}$ that satisfies the following properties:
1. $\|v\| = 0 \Rightarrow v = 0$.
2. $\|v\| \geq 0, \forall\, v \in V$.

3. $\|\alpha v\| = |\alpha|.\|v\|$ for all $\alpha \in \mathbb{K}$ and $v \in V$.
4. $\|u+v\| \leq \|u\| + \|v\| \; \forall\, u,\; v \in V$.

Remark 9

A norm in a vector space plays the same role as the absolute value in $\mathbb{R}$.

Definition 28 p-Norms

Let V be a vector space over the field F of scalars. Let $p > 0$ and $v \in V$.

The p-norm of v is defined by: $\|v\|_p = (|v_1|^p + |v_2|^p + \cdots |v_n|^p)^{\frac{1}{p}} = (\sum_{i=1}^n |v_i|^p)^{\frac{1}{p}}$

Example 33

For $p = 1 \Longrightarrow \|v\|_1 = (|v_1|^1 + |v_2|^1 + \cdots |v_n|^1)^{\frac{1}{1}} = \sum_{i=1}^n |v_i|$

For $p = 5 \Longrightarrow \|v\|_1 = (|v_1|^5 + |v_2|^5 + \cdots |v_n|^5)^{\frac{1}{5}}$

Definition 29 $1, 2$ and ∞ Norms

Let V be a vector space over the field F of scalars.

The following three norms are the most commonly used in practice:

1. 1-Norm (Manhattan Norm or Taxicab Norm)

$\|v\|_1 = \sum_{i=1}^n |v_i|$.

This norm sums the absolute values of the components of the vector.

2. 2-Norm (Euclidean Norm)

$\|v\|_2 = (\sum_{i=1}^n |v_i|^2)^{\frac{1}{2}} = \sqrt{\sum_{i=1}^n |v_i|^2}$

This norm is the square root of the sum of the squares of the components, which corresponds to the Euclidean distance from the origin.

3. ∞-Norm (Maximum Norm or Chebyshev Norm)

$\|v\|_\infty = \max_i |v_i|$.

This norm is the maximum absolute value among the components of the vector.

Example 34

If $v = (3 \quad 4-3i \quad 1)$ then $\|v\|_1 = 9$ and $\|v\|_2 = \sqrt{35}$ and $\|v\|_\infty = 5$.

Matrix Norms

Definition 30 Matrix Norms

A matrix norm is a function from $\mathbb{C}^{m\times n}$ to $\mathbb{R}$ that satisfies the following properties:

1. $\|A\| = 0 \Longrightarrow A = 0$.
2. $\|A\| \geq 0 \,, \forall\, v \in V$
3. $\|\alpha A\| = |\alpha|.\|vA\|$ for all $\alpha \in \mathbb{K}$ and $v \in \mathbb{C}^{m,n}$.
4. $\|A+B\| \leq \|A\| + \|B\| \; \forall\, A,\; B \in \mathbb{C}^{m,n}$.
5. $\|AB\| \leq \|A\|.\|B\|$ if it is possible.

Definition 31 p-Norms

Let $p > 0$ and $A \in \mathbb{C}^{m\times n}$.

The p-norm of A is defined by: $\|A\|_p = \sup \frac{\|Ax\|_p}{\|x\|_p} = \max_{\|x\|_p = 1} \|Ax\|_p$.

This norm is a matrix norm known as the **subordinate matrix norm** (to the given vector norm).

Definition 32 $1, 2$ and ∞ Norms

Let A be a matrix.

The following three norms are the most commonly used in practice:

1. **1-Norm (Maximum Absolute Column Sum Norm)**

$\|A\|_1 = \max_j \sum_i |a_{ij}|$

This norm is the maximum of the sums of the absolute values of the entries in each column.

2. **2-Norm**

$\|A\|_2 = \sqrt{\lambda_{max}(A^T A)}$

3. **∞-Norm (Maximum Absolute Row Sum Norm)**

$\|A\|_\infty = \max_i \sum_j |a_{ij}|$.

This norm is the maximum of the sums of the absolute values of the entries in each row.

Example 35

If $A = \frac{1}{\sqrt{3}}\begin{pmatrix} 3 & -1 \\ 0 & \sqrt{8} \end{pmatrix}$ then $\|A\|_1 = \frac{1}{\sqrt{3}} + \frac{\sqrt{8}}{\sqrt{3}}$ and $\|A\|_2 = 2$ and $\|A\|_\infty = \frac{4}{\sqrt{3}}$.

Definition 33 Frobenius Norm

Let A be a matrix.

The Frobenius matrix norm is defined by: $\|A\|_F = \sqrt{tr(A^T A)} = \sqrt{\sum_{i,j} |a_{i,j}|^2}$.

This norm is analogous to the 2-norm for vectors but applied to matrices. It is the square root of the sum of the absolute squares of the matrix elements.

Example 36

If $A = \frac{1}{\sqrt{3}}\begin{pmatrix} 3 & -1 \\ 0 & \sqrt{8} \end{pmatrix}$ then $\|A\|_F = \sqrt{6}$

❖ Inner Product

Definition 34 Inner Product

Inner product on a vector space V is a function that associates with each pair of vectors x and y a number, satisfying the following properties:

1. $\langle x|x \rangle \geq 0$.
2. $\langle x|x \rangle = 0 \iff x = 0$.
3. $\langle x|\alpha y \rangle = \alpha \langle x|y \rangle \; \forall \alpha \in F$.
4. $\langle x|y + z \rangle = \langle x|y \rangle + \langle x|z \rangle$
5. $\langle x|y \rangle = \langle y|x \rangle$

Example 37

The following function defines a dot product: $\langle A|B \rangle = tr(A^T B)$.

Remark 10

1. Inner product is also called dot product or scalar product.
2. If V is a vector space equipped with an inner product $\langle x|y \rangle$, then $\|x\| = \sqrt{\langle x|x \rangle}$ is a norm on the space V.