Université Batna-2
Faculté des Mathématique et d'informatique
Département d'informatique
module : probabilités et Statistique ($L3$)
Corrigé **Série** n : 2 (*Variables Aléatoires*)

Exercice 1 $\Omega = \{1, 2, 3, 4, 5, 6\}$

$X(\omega) = 2\omega$ donc $X(1) = 2.1 = 2, X(2) = 2.2 = 4 \ldots\ X(6) = 2.6 = 12$ alors $X(\Omega) = \{2, 4, 6, 8, 10, 12\}$
Calculons f_X la loi de X ;
$f_X(2) = P(X = 2) = P(\{\omega \in \Omega, X(\omega) = 2\}) = P(\{1\}) = \frac{1}{6}$
$f_X(4) = P(X = 4) = P(\{\omega \in \Omega, X(\omega) = 4\}) = P(\{2\}) = \frac{1}{6}$
de mem façon on calcule $f_X(6), f_X(8), f_X(10)$ et $f_X(12)$. Finalement

x_i	2	4	6	8	10	12
$f_X(x_i)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

Calculons f_Y la loi de Y ;
$Y(1) = Y(3) = Y(5) = 1$ et $Y(2) = Y(4) = Y(6) = 3$ donc $Y(\Omega) = \{1, 3\}$
$f_Y(1) = P(Y = 1) = P(\{\omega \in \Omega, Y(\omega) = 1\}) = P(\{1, 3, 5\}) = \frac{3}{6}$
$f_Y(3) = P(Y = 3) = P(\{\omega \in \Omega, Y(\omega) = 3\}) = P(\{2, 4, 6\}) = \frac{3}{6}$

y_i	1	3
$f_Y(y_i)$	$\frac{3}{6}$	$\frac{3}{6}$

$(X + Y)(\omega) = X(\omega) + Y(\omega)$ exemple $(X + Y)(1) = X(1) + Y(1) = 2 + 1 = 3$ de meme manière on trouve $(X + Y)(\Omega) = \{3, 7, 11, 15\}$ et la loi de $X + Y$ et f_{X+Y} définie par le tableau suivant

z_i	3	7	11	15
$f_{X+Y}(z_i)$	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{2}{6}$	$\frac{1}{6}$

Les fonctions de répartition ; $F_X(x)$

x_i	2	4	6	8	10	12
$f_X(x_i)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$
$F_X(x_i)$	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	$\frac{6}{6} = 1$

$F_Y(y)$

y_i	1	3
$f_Y(y_i)$	$\frac{3}{6}$	$\frac{3}{6}$
$F_Y(y_i)$	$\frac{3}{6}$	$\frac{6}{6} = 1$

$F_{X+Y}(z)$

z_i	3	7	11	15
$f_{X+Y}(z_i)$	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{2}{6}$	$\frac{1}{6}$
$F_{X+Y}(z)$	$\frac{1}{6}$	$\frac{3}{6}$	$\frac{5}{6}$	1

Calculons $E(X), Var(X)$ et σ_X
$E(X) = \sum x_i f_X(x_i) = 2.\frac{1}{6} + 4.\frac{1}{6} + 6.\frac{1}{6} + 8.\frac{1}{6} + 10.\frac{1}{6} + 12.\frac{1}{6} = 7$
$Var(X) = E(X^2) - E(X)^2$
$E(X^2) = \sum x_i^2 f_X(x_i) = 2^2.\frac{1}{6} + 4^2.\frac{1}{6} + 6^2.\frac{1}{6} + 8^2.\frac{1}{6} + 10^2.\frac{1}{6} + 12^2.\frac{1}{6} = 60.667$
$Var(X) = 60,66 - 49 = 11,66$
$\sigma_X = \sqrt{11,66} = 3,41$
De mem façons on calcule $E(Y)$, $E(X + Y)$, $Var(Y)$, $Var(X + Y)$, σ_Y et σ_{X+Y}.

Exercice 2 $\Omega = \{FFF, FFP, FPF, PFF, FPP, PFP, PPF, PPP\}$

$P(FFF) = \frac{3}{4} \cdot \frac{3}{4} \cdot \frac{3}{4} = \frac{27}{64}$; $P(FFP) = P(FPF) = P(PFF) = \frac{3}{4} \cdot \frac{3}{4} \cdot \frac{1}{4} = \frac{9}{64}$;
$P(FPP) = P(PFP) = P(FPP) = \frac{3}{4} \cdot \frac{1}{4} \cdot \frac{1}{4} = \frac{3}{64}$; $P(PPP) = \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{64}$
$X(FFF) = 3, X(FFP) = 2, X(FPF) = 1, X(PFF) = 2, X(FPP) = 1, X(PFP) = 1, X(PPF) = 1, X(PPP) = 0$
$X(\Omega) = \{0, 1, 2, 3\}$
$f(x)$ la loi de X et la fonction de répartition $F(x)$
$f(0) = P(X = 0) = P(\{\omega \in \Omega, X(\omega) = 0\}) = P(\{PPP\}) = \frac{1}{64}$
$f(1) = P(X = 1) = P(\{\omega \in \Omega, X(\omega) = 1\}) = P(\{FPP, PFP, PPF, FPF\}) = \frac{18}{64}$
....

x_i	0	1	2	3
$f(x_i)$	$\frac{1}{64}$	$\frac{18}{64}$	$\frac{18}{64}$	$\frac{27}{64}$
$F(x_i)$	$\frac{1}{64}$	$\frac{19}{64}$	$\frac{37}{64}$	1

$E(X) = \sum x_i f_X(x_i) = 0.\frac{1}{64} + 1.\frac{18}{64} + 2.\frac{18}{64} + 3.\frac{27}{64} = 2.10$
$Var(X) = E(X^2) - E(X)^2$
$E(X^2) = \sum x_i^2 f_X(x_i) = 0^2.\frac{1}{64} + 1^2.\frac{18}{64} + 2^2.\frac{18}{64} + 3^2.\frac{27}{64} = \frac{333}{64} = 5.20$
$Var(X) = 5.2 - 4.41 = 0.79$
$\sigma_X = \sqrt{0.79} = 0.88882$

Exercice 3 $P(1 \leq X \leq 1,5) = \int_1^{1.5} f(x)\,dx = \int_1^{1.5} \frac{1}{2}x\ dx = \left[\frac{1}{4}x^2\right]_1^{1.5} = \frac{5}{16}$

$E(X) = \int_{IR} xf(x)\,dx = \int_0^2 x.\frac{1}{2}xdx = \int_0^2 \frac{1}{2}x^2dx = \left[\frac{1}{6}x^3\right]_0^2 = \frac{4}{3}$
$Var(X) = E(X^2) - E(X)^2$
$E(X^2) = \int_{IR} x^2 f(x)\,dx = \int_0^2 \frac{1}{2}x^3dx = \left[\frac{1}{8}x^4\right]_0^2 = 2$
$Var(X) = 2 - \left(\frac{5}{16}\right)^2 = 1.9023$
La fonction de répartition $F(x) = P(X \leq x) = \int_{-\infty}^x f(t)\,dt$
donc $F(x) = \left\{ \begin{array}{c} \int_{-\infty}^x 0dt = 0 \text{ si } x<0 \\ \int_0^x \frac{1}{2}tdt = \frac{1}{4}x^2 \text{ si } 0 \leq x \leq 2 \\ 1 \text{ si } x \geq 2 \end{array} \right\}$
alors $F(x) = \left\{ \begin{array}{c} 0 \text{ si } x<0 \\ \frac{1}{4}x^2 \text{ si } 0 \leq x \leq 2 \\ 1 \text{ si } x \geq 2 \end{array} \right\}$

Exercice 4 Trouvons k

On a $\int_{IR} f(x)\,dx = 1$ alors $\int_0^3 \left(\frac{1}{6}x + k\right) dx = 1$ implique $\left[\frac{1}{12}x^2 + kx\right]_0^3 = 1$ implique $\frac{9}{12} + 3k = 1$ implique $k = \frac{1}{12}$
finalement la densité devient $f(x) = \left\{\begin{array}{c} \frac{1}{6}x + \frac{1}{12} \text{ si } 0 \leq x \leq 3 \\ 0 \text{ ailleurs} \end{array}\right\}$
$P(1 \leq X \leq 2) = \int_1^2 \left(\frac{1}{6}x + \frac{1}{12}\right) dx = \left[\frac{1}{12}x^2 + \frac{1}{12}x\right]_1^2 = \frac{1}{3}$

$E(X) = \int_{IR} xf(x)\,dx = \int_0^3 x\left(\frac{1}{6}x + \frac{1}{12}\right) dx = \int_0^3 \left(\frac{1}{6}x^2 + \frac{1}{12}x\right) dx = ...$
$E(X^2) = \int_{IR} x^2 f(x)\,dx = \int_0^3 x^2\left(\frac{1}{6}x + \frac{1}{12}\right) dx = \int_0^3 \left(\frac{1}{6}x^3 + \frac{1}{12}x^2\right) dx = ...$
$Var(X) = E(X^2) - E(X)^2$
$F(x) = P(X \leq x) = \int_{-\infty}^x f(t)\,dt$
$F(x) = \left\{ \begin{array}{c} \int_{-\infty}^x 0dt = 0 \text{ si } x<0 \\ \int_0^x \left(\frac{1}{6}t + \frac{1}{12}\right) dt = ... \text{ si } 0 < x \leq 3 \\ 1 \text{ si } x \geq 3 \end{array} \right\}$